Analogues of the adjoint matrix for generalized inverses and corresponding Cramer rules.

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Abstract

In this article, we introduce determinantal representations of the Moore - Penrose inverse and the Drazin inverse which are based on analogues of the classical adjoint matrix. Using the obtained analogues of the adjoint matrix, we get Cramer rules for the least squares solution and for the Drazin inverse solution of singular linear systems. Finally, determinantal expressions for $\mathbf{A}^{+}\mathbf{A}$, $\mathbf{A}\mathbf{A}^{+}$, and $\mathbf{A}^{D}\mathbf{A}$ are presented.

Keywords: Moore-Penrose inverse; Drazin inverse; system of linear equations; least squares solution; Cramer rule
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1 Introduction

Determinantal representation of the Moore - Penrose inverse was studied in [1, 2, 8, 11, 12]. The main result consists in the following theorem.

Theorem 1.1 *The Moore - Penrose inverse $\mathbf{A}^{+} = (a_{ij}^{+})_{n\times m}$ of $\mathbf{A} \in \mathbb{C}_r^{m\times n}$ has the following determinantal representation*

$$a_{ij}^{+} = \frac{\sum\limits_{(\alpha,\,\beta)\in N_r\{j,\,i\}} \left|(\mathbf{A}^*)_{\alpha}^{\beta}\right| \frac{\partial}{\partial a_{j\,i}} \left|\mathbf{A}_{\beta}^{\alpha}\right|}{\sum\limits_{(\gamma,\,\delta)\in N_r} \left|(\mathbf{A}^*)_{\gamma}^{\delta}\right| \left|\mathbf{A}_{\delta}^{\gamma}\right|},\ 1 \le i, j \le n.$$

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Stanimirovic' [13] introduced a determinantal representation of the Drazin inverse by the following theorem.

Theorem 1.2 *The Drazin inverse* $\mathbf{A}^D = \left(a_{ij}^D\right)$ *of an arbitrary matrix* $\mathbf{A} \in \mathbb{C}^{n\times n}$ *with* $Ind\mathbf{A} = k$ *possesses the following determinantal representation*

$$a_{ij}^D = \frac{\sum\limits_{(\alpha,\,\beta)\in N_{r_k}\{j,\,i\}} \left|(\mathbf{A}^s)_\alpha^\beta\right| \frac{\partial}{\partial a_{j\,i}} \left|\mathbf{A}_\beta^\alpha\right|}{\sum\limits_{(\gamma,\,\delta)\in N_{r_k}} \left|(\mathbf{A}^s)_\gamma^\delta\right| \left|\mathbf{A}_\delta^\gamma\right|},\ 1 \leq i, j \leq n;$$

where $s \geq k$ *and* $r_k = rank\mathbf{A}^s$*.*

These determinantal representations of generalized inverses are based on corresponding full-rank representations.

We use the following notations from [1, 12]. Let $\mathbb{C}^{m\times n}$ be the set of m by n matrices with complex entries, $\mathbb{C}_r^{m\times n}$ be the subset of $\mathbb{C}^{m\times n}$ in which every matrix has rank r. $\mathbf{I}_m$ denotes the identity matrix of order m, and $\|.\| = \|.\|_2$ is the Euclidean vector norm. Let $\alpha := \{\alpha_1, \ldots, \alpha_k\} \subseteq \{1, \ldots, m\}$ and $\beta := \{\beta_1, \ldots, \beta_k\} \subseteq \{1, \ldots, n\}$ be subsets of the order $1 \leq k \leq \min\{m, n\}$. Then $\left|\mathbf{A}_\beta^\alpha\right|$ denotes the minor of $\mathbf{A} \in \mathbb{C}^{m\times n}$ determined by the rows indexed by α and the columns indexed by β. Clearly, $|\mathbf{A}_\alpha^\alpha|$ denotes a principal minor determined by the rows and columns indexed by α. The cofactor of a_{ij} in $\mathbf{A} \in \mathbb{C}^{n\times n}$ is denoted by $\frac{\partial}{\partial a_{ij}}|\mathbf{A}|$. For $1 \leq k \leq n$, denote by $L_{k,n} := \{\,\alpha : \alpha = (\alpha_1, \ldots, \alpha_k)\,,\ 1 \leq \alpha_1 \leq \ldots \leq \alpha_k \leq n\}$ the collection of strictly increasing sequences of k integers chosen from $\{1, \ldots, n\}$. Let $N_k := L_{k,m} \times L_{k,n}$. For fixed $\alpha \in L_{p,m}$, $\beta \in L_{p,n}$, $1 \leq p \leq k$, let

$$\begin{gathered} I_{k,\,m}\left(\alpha\right) := \{I :\ I \in L_{k,\,m}, I \supseteq \alpha\}, \\ J_{k,\,n}\left(\beta\right) := \{J :\ J \in L_{k,\,n}, J \supseteq \beta\}, \\ N_k\left(\alpha, \beta\right) := I_{k,\,m}\left(\alpha\right) \times J_{k,\,n}\left(\beta\right) \end{gathered}$$

For case $i \in \alpha$ and $j \in \beta$, we denote

$$\begin{gathered} I_{k,m}\{i\} := \{\alpha :\ \alpha \in L_{k,m}, i \in \alpha\}, J_{k,\,n}\{j\} := \{\beta :\ \beta \in L_{k,n}, j \in \beta\}, \\ N_k\{i, j\} := I_{k,\,m}\{i\} \times J_{k,\,n}\{j\}. \end{gathered}$$

In this paper we introduce determinantal representations of the Moore - Penrose inverse and of the Drazin inverse based on corresponding limit representations. The obtained determinantal representations can be considered as founded on some analogues of the classical adjoint matrix. The corresponding Cramer rules for the complex system of linear equations with a rectangular or singular coefficient matrix follow from these analogues.

2 Analogues of the classical adjoint matrix for the Moore - Penrose inverse

We shall use the following well-known facts (see, for example, [9]).

Definition 2.1 *The matrix* $\mathbf{A}^{+} \in \mathbb{C}^{n\times m}$ *is called the Moore - Penrose inverse of an arbitrary* $\mathbf{A} \in \mathbb{C}^{m\times n}$ *if it satisfies the equations*

$$\mathbf{A}\mathbf{A}^{+}\mathbf{A} = \mathbf{A};\;\; \mathbf{A}^{+}\mathbf{A}\mathbf{A}^{+} = \mathbf{A}^{+};\;\; (\mathbf{A}\mathbf{A}^{+})^{*} = \mathbf{A}\mathbf{A}^{+};\;\; (\mathbf{A}^{+}\mathbf{A})^{*} = \mathbf{A}^{+}\mathbf{A}.$$

The superscript $*$ denotes conjugate transpose matrix.

Lemma 2.1 *[9] There exists a unique Moore - Penrose inverse* $\mathbf{A}^{+}$ *of* $\mathbf{A} \in \mathbb{C}^{m\times n}$.

Lemma 2.2 *[9] If* $\mathbf{A} \in \mathbb{C}^{m\times n}$, *then*

$$\mathbf{A}^{+} = \lim_{\lambda\to 0}\mathbf{A}^{*}\left(\mathbf{A}\mathbf{A}^{*} + \lambda\mathbf{I}\right)^{-1} = \lim_{\lambda\to 0}\left(\mathbf{A}^{*}\mathbf{A} + \lambda\mathbf{I}\right)^{-1}\mathbf{A}^{*},$$

where $\lambda \in \mathbb{R}_{+}$, *and* $\mathbb{R}_{+}$ *is the set of positive real numbers.*

Lemma 2.3 *[9] If* $\mathbf{A} \in \mathbb{C}^{m\times n}$, *then the following statements are true.*

i) If $\operatorname{rank}\mathbf{A} = \mathrm{n}$, *then* $\mathbf{A}^{+} = \left(\mathbf{A}^{*}\mathbf{A}\right)^{-1}\mathbf{A}^{*}$.

ii) If $\operatorname{rank}\mathbf{A} = \mathrm{m}$, *then* $\mathbf{A}^{+} = \mathbf{A}^{*}\left(\mathbf{A}\mathbf{A}^{*}\right)^{-1}$.

iii) If $\operatorname{rank}\mathbf{A} = \mathrm{n} = \mathrm{m}$, *then* $\mathbf{A}^{+} = \mathbf{A}^{-1}$.

Theorem 2.1 *[9] Let* d_r *be the sum of principal minors of order* r *of* $\mathbf{A} \in \mathbb{C}^{n\times n}$. *Then its characteristic polynomial* $p_{\mathbf{A}}(t)$ *can be expressed as* $p_{\mathbf{A}}(t) = \det(t\mathbf{I} - \mathbf{A}) = t^{n} - d_1 t^{n-1} + d_2 t^{n-2} - \ldots + (-1)^{n} d_n$.

Denote by $\mathbf{a}_{.j}$ and $\mathbf{a}_{i.}$ the jth column and the ith row of $\mathbf{A} \in \mathbb{C}^{m\times n}$ respectively. In the same way, denote by $\mathbf{a}^{*}_{.j}$ and $\mathbf{a}^{*}_{i.}$ the jth column and the ith row of Hermitian adjoint matrix $\mathbf{A}^{*}$. Let $\mathbf{A}_{.j}(\mathbf{b})$ denote the matrix obtained from $\mathbf{A}$ by replacing its jth column with some vector $\mathbf{b}$, and let $\mathbf{A}_{i.}(\mathbf{b})$ denote the matrix obtained from $\mathbf{A}$ by replacing its ith row with $\mathbf{b}$.

Lemma 2.4 *If* $\mathbf{A} \in \mathbb{C}^{m\times n}_{r}$, *then* $\operatorname{rank}\left(\mathbf{A}^{*}\mathbf{A}\right)_{.i}\left(\mathbf{a}^{*}_{.j}\right) \le r$.

Proof. Let $\mathbf{P}_{i\,k}(-a_{j\,k}) \in \mathbb{C}^{n\times n}$, $(k \neq i)$, be the matrix with $-a_{j\,k}$ in the (i,k) entry, 1 in all diagonal entries, and 0 in others. It is the matrix of an elementary transformation. It follows that

$$(\mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{a}^*_{.j}\right)\cdot\prod_{k\neq i}\mathbf{P}_{i\,k}(-a_{j\,k}) = \begin{array}{c}\left(\begin{array}{ccccc} \sum\limits_{k\neq j} a^*_{1k}a_{k1} & \dots & a^*_{1j} & \dots & \sum\limits_{k\neq j} a^*_{1k}a_{kn} \\ \dots & \dots & \dots & \dots & \dots \\ \sum\limits_{k\neq j} a^*_{nk}a_{k1} & \dots & a^*_{nj} & \dots & \sum\limits_{k\neq j} a^*_{nk}a_{kn} \end{array}\right) \\ \begin{array}{ccccc} \phantom{\sum a^*_{1k}a_{k1}} & & i-th & & \phantom{\sum a^*_{1k}a_{kn}} \end{array}\end{array}.$$

The obtained above matrix has the following factorization.

$$\begin{array}{c}\left(\begin{array}{ccccc} \sum\limits_{k\neq j} a^*_{1k}a_{k1} & \dots & a^*_{1j} & \dots & \sum\limits_{k\neq j} a^*_{1k}a_{kn} \\ \dots & \dots & \dots & \dots & \dots \\ \sum\limits_{k\neq j} a^*_{nk}a_{k1} & \dots & a^*_{nj} & \dots & \sum\limits_{k\neq j} a^*_{nk}a_{kn} \end{array}\right) \\ \begin{array}{ccccc} \phantom{\sum a^*_{1k}a_{k1}} & & i-th & & \phantom{\sum a^*_{1k}a_{kn}} \end{array}\end{array} =$$

$$= \begin{pmatrix} a^*_{11} & a^*_{12} & \dots & a^*_{1m} \\ a^*_{21} & a^*_{22} & \dots & a^*_{2m} \\ \dots & \dots & \dots & \dots \\ a^*_{n1} & a^*_{n2} & \dots & a^*_{nm} \end{pmatrix} \begin{array}{c}\left(\begin{array}{ccccc} a_{11} & \dots & 0 & \dots & a_{n1} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & \dots & 0 & \dots & a_{mn} \end{array}\right) \\ \begin{array}{ccccc} \phantom{a_{m1}} & & i-th & & \phantom{a_{mn}} \end{array}\end{array} j-th.$$

Denote by $\tilde{\mathbf{A}} := \begin{array}{c}\left(\begin{array}{ccccc} a_{11} & \dots & 0 & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & \dots & 0 & \dots & a_{mn} \end{array}\right) \\ \begin{array}{ccccc} \phantom{a_{m1}} & & i-th & & \phantom{a_{mn}} \end{array}\end{array} j-th$. The matrix $\tilde{\mathbf{A}}$ is obtained from $\mathbf{A}$ by replacing all entries of the jth row and of the ith column with zeroes except that the (j,i) entry equals 1. Elementary transformations of a matrix do not change its rank. It follows that $\operatorname{rank}(\mathbf{A}^*\mathbf{A})_{.\,i}\left(\mathbf{a}^*_{.j}\right) \leq \min\left\{\operatorname{rank}\mathbf{A}^*, \operatorname{rank}\tilde{\mathbf{A}}\right\}$. Since $\operatorname{rank}\tilde{\mathbf{A}} \geq \operatorname{rank}\mathbf{A} = \operatorname{rank}\mathbf{A}^*$ and $\operatorname{rank}\mathbf{A}^*\mathbf{A} = \operatorname{rank}\mathbf{A}$ the proof is completed.$\blacksquare$

The following lemma is proved in the same way.

Lemma 2.5 *If* $\mathbf{A} \in \mathbb{C}^{m\times n}_r$, *then* $\operatorname{rank}(\mathbf{A}\mathbf{A}^*)_{i\,.}\left(\mathbf{a}^*_{j\,.}\right) \leq r$.

Theorem 2.2 *The Moore-Penrose inverse* $\mathbf{A}^{+}$ *of* $\mathbf{A} \in \mathbb{C}_r^{m\times n}$ *can be represented as follows*

$$\begin{array}{l} \mathbf{A}^{+} = \left(\frac{l_{ij}}{d_r(\mathbf{A}^*\mathbf{A})}\right)_{n\times m}, \textit{ where} \\ l_{ij} = \sum\limits_{\beta\in J_{r,\,n}\{i\}} \left|\left((\mathbf{A}^*\mathbf{A})_{.\,i}(\mathbf{a}^*_{.j})\right)^{\beta}_{\beta}\right|, \;\; d_r(\mathbf{A}^*\mathbf{A}) = \sum\limits_{\beta\in J_{r,\,n}} \left|(\mathbf{A}^*\mathbf{A})^{\beta}_{\beta}\right|, \end{array} \tag{1}$$

or

$$\begin{array}{l} \mathbf{A}^{+} = \left(\frac{r_{ij}}{d_r(\mathbf{A}\mathbf{A}^*)}\right)_{n\times m}, \textit{ where} \\ r_{ij} = \sum\limits_{\alpha\in I_{r,\,m}\{j\}} \left|((\mathbf{A}\mathbf{A}^*)_{j\,.}(\mathbf{a}^*_{i\,.}))^{\alpha}_{\alpha}\right|, \;\; d_r(\mathbf{A}\mathbf{A}^*) = \sum\limits_{\alpha\in I_{r,\,m}} |(\mathbf{A}\mathbf{A}^*)^{\alpha}_{\alpha}|. \end{array} \tag{2}$$

Proof. At first we shall obtain the representation (1). If $\lambda \in \mathbb{R}_+$, then the matrix $(\lambda\mathbf{I} + \mathbf{A}^*\mathbf{A}) \in \mathbb{C}^{n\times n}$ is Hermitian and $\operatorname{rank}(\lambda\mathbf{I} + \mathbf{A}^*\mathbf{A}) = n$. Hence, there exists its inverse

$$(\lambda\mathbf{I} + \mathbf{A}^*\mathbf{A})^{-1} = \frac{1}{\det(\lambda\mathbf{I} + \mathbf{A}^*\mathbf{A})} \begin{pmatrix} L_{11} & L_{21} & \dots & L_{n\,1} \\ L_{12} & L_{22} & \dots & L_{n\,2} \\ \dots & \dots & \dots & \dots \\ L_{1\,n} & L_{2\,n} & \dots & L_{n\,n} \end{pmatrix},$$

where L_{ij} $(\forall i, j = \overline{1, n})$ is a cofactor in $\lambda\mathbf{I} + \mathbf{A}^*\mathbf{A}$. By Lemma 2.2, $\mathbf{A}^{+} = \lim\limits_{\lambda\to 0} (\lambda\mathbf{I} + \mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^*$, so that

$$\mathbf{A}^{+} = \lim_{\lambda\to 0} \begin{pmatrix} \frac{\det(\lambda\mathbf{I}+\mathbf{A}^*\mathbf{A})_{.1}\left(\mathbf{a}^*_{.1}\right)}{\det(\lambda\mathbf{I}+\mathbf{A}^*\mathbf{A})} & \dots & \frac{\det(\lambda\mathbf{I}+\mathbf{A}^*\mathbf{A})_{.\,1}(\mathbf{a}^*_{.\,m})}{\det(\lambda\mathbf{I}+\mathbf{A}^*\mathbf{A})} \\ \dots & \dots & \dots \\ \frac{\det(\lambda\mathbf{I}+\mathbf{A}^*\mathbf{A})_{.\,n}\left(\mathbf{a}^*_{.1}\right)}{\det(\lambda\mathbf{I}+\mathbf{A}^*\mathbf{A})} & \dots & \frac{\det(\lambda\mathbf{I}+\mathbf{A}^*\mathbf{A})_{.\,n}(\mathbf{a}^*_{.\,m})}{\det(\lambda\mathbf{I}+\mathbf{A}^*\mathbf{A})} \end{pmatrix}. \tag{3}$$

From Theorem 2.1 we get

$$\det(\lambda\mathbf{I} + \mathbf{A}^*\mathbf{A}) = \lambda^n + d_1\lambda^{n-1} + d_2\lambda^{n-2} + \ldots + d_n,$$

where d_r $(\forall r = \overline{1, n-1})$ is a sum of principal minors of $\mathbf{A}^*\mathbf{A}$ of order r and $d_n = \det\mathbf{A}^*\mathbf{A}$. Since $\operatorname{rank}\mathbf{A}^*\mathbf{A} = \operatorname{rank}\mathbf{A} = r$, then $d_n = d_{n-1} = \ldots = d_{r+1} = 0$ and

$$\det(\lambda\mathbf{I} + \mathbf{A}^*\mathbf{A}) = \lambda^n + d_1\lambda^{n-1} + d_2\lambda^{n-2} + \ldots + d_r\lambda^{n-r}. \tag{4}$$

In the same way, we have for arbitrary $1 \leq i \leq n$ and $1 \leq j \leq m$ from Theorem 2.1

$$\det\left(\lambda\mathbf{I} + \mathbf{A}^*\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^*_{.j}\right) = l_1^{(ij)}\lambda^{n-1} + l_2^{(ij)}\lambda^{n-2} + \ldots + l_n^{(ij)},$$

where for an arbitrary $1 \leq k \leq n-1$, $l_k^{(ij)} = \sum\limits_{\beta \in J_{k,\,n}\{i\}} \left|\left((\mathbf{A}^*\mathbf{A})_{.\,i}(\mathbf{a}^*_{.j})\right)^{\beta}_{\beta}\right|$, and $l_n^{(i\,j)} = \det\left(\mathbf{A}^*\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^*_{.j}\right)$. By Lemma 2.4, $\operatorname{rank}\left(\mathbf{A}^*\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^*_{.j}\right) \leq r$ so that if $k > r$, then $\left|\left((\mathbf{A}^*\mathbf{A})_{.\,i}(\mathbf{a}^*_{.j})\right)^{\beta}_{\beta}\right| = 0$, $(\forall\beta \in J_{k,n}\{i\}, \forall i = \overline{1,n}, \forall j = \overline{1,m})$. Therefore if $r+1 \leq k < n$, then $l_k^{(ij)} = \sum\limits_{\beta \in J_{k,\,n}\{i\}} \left|\left((\mathbf{A}^*\mathbf{A})_{.\,i}(\mathbf{a}^*_{.j})\right)^{\beta}_{\beta}\right| = 0$ and $l_n^{(i\,j)} = \det\left(\mathbf{A}^*\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^*_{.j}\right) = 0$, $\left(\forall i = \overline{1,n},\ \forall j = \overline{1,m}\right)$. Finally we obtain

$$\det\left(\lambda\mathbf{I} + \mathbf{A}^*\mathbf{A}\right)_{.\,i}\left(\mathbf{a}^*_{.j}\right) = l_1^{(i\,j)}\lambda^{n-1} + l_2^{(i\,j)}\lambda^{n-2} + \ldots + l_r^{(ij)}\lambda^{n-r}. \tag{5}$$

By replacing the denominators and the numerators of the fractions in entries of matrix (3) with the expressions (4) and (5) respectively, we get

$$\mathbf{A}^+ = \lim_{\lambda\to 0}\begin{pmatrix} \frac{l_1^{(11)}\lambda^{n-1}+\ldots+l_r^{(11)}\lambda^{n-r}}{\lambda^n+d_1\lambda^{n-1}+\ldots+d_r\lambda^{n-r}} & \ldots & \frac{l_1^{(1m)}\lambda^{n-1}+\ldots+l_r^{(1m)}\lambda^{n-r}}{\lambda^n+d_1\lambda^{n-1}+\ldots+d_r\lambda^{n-r}} \\ \ldots & \ldots & \ldots \\ \frac{l_1^{(n1)}\lambda^{n-1}+\ldots+l_r^{(n1)}\lambda^{n-r}}{\lambda^n+d_1\lambda^{n-1}+\ldots+d_r\lambda^{n-r}} & \ldots & \frac{l_1^{(nm)}\lambda^{n-1}+\ldots+l_r^{(nm)}\lambda^{n-r}}{\lambda^n+d_1\lambda^{n-1}+\ldots+d_r\lambda^{n-r}} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{l_r^{(11)}}{d_r} & \ldots & \frac{l_r^{(1m)}}{d_r} \\ \ldots & \ldots & \ldots \\ \frac{l_r^{(n1)}}{d_r} & \ldots & \frac{l_r^{(nm)}}{d_r} \end{pmatrix}.$$

From here the representation (1) of $\mathbf{A}^+$ follows by denoting $l_r^{(ij)} = l_{ij}$.

We obtain the representation (2) in the same way. ■

Remark 2.1 *If* $\operatorname{rank}\mathbf{A} = n$*, then from Lemma 2.3 we get* $\mathbf{A}^+ = \left(\mathbf{A}^*\mathbf{A}\right)^{-1}\mathbf{A}^*$*. Representing* $\left(\mathbf{A}^*\mathbf{A}\right)^{-1}$ *by the classical adjoint matrix, we have*

$$\mathbf{A}^+ = \frac{1}{\det(\mathbf{A}^*\mathbf{A})}\begin{pmatrix} \det(\mathbf{A}^*\mathbf{A})_{.1}\left(\mathbf{a}^*_{.1}\right) & \ldots & \det(\mathbf{A}^*\mathbf{A})_{.1}\left(\mathbf{a}^*_{.m}\right) \\ \ldots & \ldots & \ldots \\ \det(\mathbf{A}^*\mathbf{A})_{.\,n}\left(\mathbf{a}^*_{.1}\right) & \ldots & \det(\mathbf{A}^*\mathbf{A})_{.\,n}\left(\mathbf{a}^*_{.m}\right) \end{pmatrix}. \tag{6}$$

If $n < m$*, then (2) is valid.*

Remark 2.2 *As above, if* $\operatorname{rank}\mathbf{A} = m$*, then*

$$\mathbf{A}^{+} = \frac{1}{\det(\mathbf{A}\mathbf{A}^{*})}\begin{pmatrix} \det(\mathbf{A}\mathbf{A}^{*})_{1\,.}\left(\mathbf{a}^{*}_{1\,.}\right) & \dots & \det(\mathbf{A}\mathbf{A}^{*})_{m\,.}\left(\mathbf{a}^{*}_{1\,.}\right) \\ \dots & \dots & \dots \\ \det(\mathbf{A}\mathbf{A}^{*})_{1\,.}\left(\mathbf{a}^{*}_{n\,.}\right) & \dots & \det(\mathbf{A}\mathbf{A}^{*})_{m\,.}\left(\mathbf{a}^{*}_{n\,.}\right) \end{pmatrix}. \qquad (7)$$

If $n > m$*, then (1) is valid as well.*

Remark 2.3 *The representation (1) can be obtained from Theorem 1.1 by using the Binet-Cauchy formula. We use another method, which is the same for the determinantal representations of the Moore-Penrose inverse by (1) and (2), and of the Drazin inverse by (11).*

Remark 2.4 *To obtain an entry of* $\mathbf{A}^{+}$ *by Theorem 1.1 one calculates* $(C_n^r C_m^r + C_{n-1}^{r-1} C_{m-1}^{r-1})$ *determinants of order* r*. Whereas by (1) we calculate as much as* $(C_n^r + C_{n-1}^{r-1})$ *determinants of order* r *or we calculate the total of* $(C_m^r + C_{m-1}^{r-1})$ *determinants by (2). Therefore the calculation of entries of* $\mathbf{A}^{+}$ *by Theorem 2.2 is easier than by Theorem 1.1.*

Corollary 2.1 *If* $\mathbf{A} \in \mathbb{C}_r^{m\times n}$ *and* $r < \min\{m, n\}$ *or* $r = m < n$*, then the projection matrix* $\mathbf{P} = \mathbf{A}^{+}\mathbf{A}$ *can be represented as*

$$\mathbf{P} = \left(\frac{p_{ij}}{d_r\left(\mathbf{A}^{*}\mathbf{A}\right)}\right)_{n\times n},$$

where $\mathbf{d}_{.\,j}$ *denotes the* j*th column of* $(\mathbf{A}^{*}\mathbf{A})$ *and, for arbitrary* $1 \le i, j \le n$*,* $p_{ij} = \sum\limits_{\beta \in J_{r,n}\{i\}} \left|\left(\left(\mathbf{A}^{*}\mathbf{A}\right)_{.\,i}(\mathbf{d}_{.j})\right)_{\beta}^{\beta}\right|$.

Proof. Representing the Moore - Penrose inverse $\mathbf{A}^{+}$ by (1), we obtain

$$\mathbf{P} = \frac{1}{d_r\left(\mathbf{A}^{*}\mathbf{A}\right)}\begin{pmatrix} l_{11} & l_{12} & \dots & l_{1m} \\ l_{21} & l_{22} & \dots & l_{2m} \\ \dots & \dots & \dots & \dots \\ l_{n1} & l_{n2} & \dots & l_{nm} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m\,1} & a_{m\,2} & \dots & a_{m\,n} \end{pmatrix}.$$

Therefore, for arbitrary $1 \le i, j \le n$ we get

$$p_{i\,j} = \sum_{k}\sum_{\beta \in J_{r,\,n}\{i\}} \left|\left((\mathbf{A}^{*}\mathbf{A})_{.\,i}(\mathbf{a}^{*}_{.\,k})\right)_{\beta}^{\beta}\right| \cdot a_{k\,j} =$$

$$= \sum_{\beta \in J_{r,\,n}\{i\}}\sum_{k} \left|\left((\mathbf{A}^{*}\mathbf{A})_{.\,i}(\mathbf{a}^{*}_{.\,k} \cdot a_{k\,j})\right)_{\beta}^{\beta}\right| = \sum_{\beta \in J_{r,\,n}\{i\}} \left|\left((\mathbf{A}^{*}\mathbf{A})_{.\,i}(\mathbf{d}^{*}_{.j})\right)_{\beta}^{\beta}\right|. \blacksquare$$

Using the representation (2) of the Moore - Penrose inverse the following corollary can be proved in the same way.

Corollary 2.2 *If* $\mathbf{A} \in \mathbb{C}_r^{m\times n}$*, where* $r < \min\{m,n\}$ *or* $r = n < m$*, then a projection matrix* $\mathbf{Q} = \mathbf{A}\mathbf{A}^+$ *can be represented as*

$$\mathbf{Q} = \left(\frac{q_{ij}}{d_r\left(\mathbf{A}\mathbf{A}^*\right)} \right)_{m\times m},$$

where $\mathbf{g}_{i.}$ *denotes the ith row of* $(\mathbf{A}\mathbf{A}^*)$ *and, for arbitrary* $1 \le i,j \le m$*,* $q_{ij} = \sum\limits_{\alpha\in I_{r,m}\{j\}} \left| \left((\mathbf{A}\mathbf{A}^*)_{j.}\left(\mathbf{g}_{i.}\right) \right)_\alpha^\alpha \right|$*.*

Remark 2.5 *By definition of the classical adjoint* $Adj(\mathbf{A})$ *for an arbitrary invertible matrix* $\mathbf{A} \in \mathbb{C}^{n\times n}$ *one may put* $Adj(\mathbf{A})\cdot\mathbf{A} = \det\mathbf{A}\cdot\mathbf{I}_n$*. If* $\mathbf{A} \in \mathbb{C}^{m\times n}$ *and* $\operatorname{rank}\mathbf{A} = n$*, then by Lemma 2.3,* $\mathbf{A}^+\mathbf{A} = \mathbf{I}_n$*. Representing the matrix* $\mathbf{A}^+$ *by (6) as* $\mathbf{A}^+ = \frac{\mathbf{L}}{\det(\mathbf{A}^*\mathbf{A})}$*, we obtain* $\mathbf{L}\mathbf{A} = \det\left(\mathbf{A}^*\mathbf{A}\right)\cdot\mathbf{I}_n$*. This means that the matrix* $\mathbf{L}$ *is a left analogue of* $Adj(\mathbf{A})$ *, where* $\mathbf{A} \in \mathbb{C}_n^{m\times n}$*. If* $\operatorname{rank}\mathbf{A} = m$*, then by Lemma 2.3,* $\mathbf{A}\mathbf{A}^+ = \mathbf{I}_m$*. Representing the matrix* $\mathbf{A}^+$ *by (7) as* $\mathbf{A}^+ = \frac{\mathbf{R}}{\det(\mathbf{A}\mathbf{A}^*)}$*, we obtain* $\mathbf{A}\mathbf{R} = \mathbf{I}_m \cdot \det\left(\mathbf{A}\mathbf{A}^*\right)$*. This means that the matrix* $\mathbf{R}$ *is a right analogue of* $Adj(\mathbf{A})$*, where* $\mathbf{A} \in \mathbb{C}_m^{m\times n}$*.*

Remark 2.6 *If* $\mathbf{A} \in \mathbb{C}_r^{m\times n}$ *and* $r < \min\{m,n\}$*, then by (1) we have* $\mathbf{A}^+ = \frac{\mathbf{L}}{d_r(\mathbf{A}^*\mathbf{A})}$*, where* $\mathbf{L} = (l_{ij}) \in \mathbb{C}^{n\times m}$*. From Corollary 2.1 we get* $\mathbf{L}\mathbf{A} = d_r\left(\mathbf{A}^*\mathbf{A}\right)\cdot \mathbf{P}$*. The matrix* $\mathbf{P}$ *is idempotent. All eigenvalues of an idempotent matrix chose from 1 or 0 only. Thus, there exists an unitary matrix* $\mathbf{U}$ *such that* $\mathbf{L}\mathbf{A} = d_r\left(\mathbf{A}^*\mathbf{A}\right)\mathbf{U}\,\mathbf{diag}\left(1,\ldots,1,0,\ldots,0\right)\mathbf{U}^*$*, where* $\mathbf{diag}\left(1,\ldots,1,0,\ldots,0\right) \in \mathbb{C}^{n\times n}$ *is a diagonal matrix. Therefore, the matrix* $\mathbf{L}$ *can be considered as a left analogue of* $Adj(\mathbf{A})$*, where* $\mathbf{A} \in \mathbb{C}_r^{m\times n}$*.*

In the same way, if $\mathbf{A} \in \mathbb{C}_r^{m\times n}$ *and* $r < \min\{m,n\}$*, then by (2) we have* $\mathbf{A}^+ = \frac{\mathbf{R}}{d_r(\mathbf{A}\mathbf{A}^*)}$*, where* $\mathbf{R} = (r_{ij}) \in \mathbb{C}^{n\times m}$*. From Corollary 2.2 we get* $\mathbf{A}\mathbf{R} = d_r\left(\mathbf{A}\mathbf{A}^*\right)\cdot\mathbf{Q}$*. The matrix* $\mathbf{Q}$ *is idempotent. There exists an unitary matrix* $\mathbf{V}$ *such that* $\mathbf{A}\mathbf{R} = d_r\left(\mathbf{A}\mathbf{A}^*\right)\mathbf{V}\,\mathbf{diag}\left(1,\ldots,1,0,\ldots,0\right)\mathbf{V}^*$*, where* $\mathbf{diag}\left(1,\ldots,1,0,\ldots,0\right) \in \mathbb{C}^{m\times m}$*. Therefore, the matrix* $\mathbf{R}$ *can be considered as a right analogue of* $Adj(\mathbf{A})$ *in this case.*

3 An analogue of the classical adjoint matrix for the Drazin inverse

Definition 3.1 *[4, 7] Let* $\mathbf{A} \in \mathbb{C}^{n\times n}$ *with* $Ind\,\mathbf{A} = k$*, where a nonnegative integer* $Ind\,\mathbf{A} := \min\limits_{k\in N\cup\{0\}}\left\{\operatorname{rank}\mathbf{A}^{k+1} = \operatorname{rank}\mathbf{A}^k\right\}$*. Then the matrix* $\mathbf{X}$

satisfying

$$\mathbf{A}^{k+1}\mathbf{X} = \mathbf{A}^{k};\ \mathbf{XAX} = \mathbf{X};\ \mathbf{AX} = \mathbf{XA} \tag{8}$$

is called the Drazin inverse of $\mathbf{A}$ *and is denoted by* $\mathbf{X} = \mathbf{A}^{D}$. *In particular, if* $Ind\,\mathbf{A} = 1$, *then the matrix* $\mathbf{X}$ *in (8) is called the group inverse and is denoted by* $\mathbf{X} = \mathbf{A}^{\#}$.

Remark 3.1 *If* $Ind\,\mathbf{A} = 0$, *then* $\mathbf{A}$ *is nonsingular, and* $\mathbf{A}^{D} \equiv \mathbf{A}^{-1}$.

The Drazin inverse can be represented explicitly by the Jordan canonical form as follows.

Theorem 3.1 *[4] If* $\mathbf{A} \in \mathbb{C}^{n\times n}$ *with* $Ind\,\mathbf{A} = k$ *and*

$$\mathbf{A} = \mathbf{P}\begin{pmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{N} \end{pmatrix}\mathbf{P}^{-1}, \tag{9}$$

where $\mathbf{C}$ *is nonsingular,* $\operatorname{rank}\mathbf{C} = \operatorname{rank}\mathbf{A}^{k}$, *and* $\mathbf{N}$ *is nilpotent of order* k, *then*

$$\mathbf{A}^{D} = \mathbf{P}\begin{pmatrix} \mathbf{C}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{P}^{-1}.$$

We use the following theorem about the limit representation of the Drazin inverse.

Theorem 3.2 *[4] If* $\mathbf{A} \in \mathbb{C}^{n\times n}$, *then*

$$\mathbf{A}^{D} = \lim_{\lambda\to 0}\left(\lambda\mathbf{I}_{n} + \mathbf{A}^{k+1}\right)^{-1}\mathbf{A}^{k},$$

where $k = Ind\,\mathbf{A}$ *and* $\lambda \in \mathbb{R}_{+}$.

Denote by $\mathbf{a}_{.j}^{(k)}$ and $\mathbf{a}_{i.}^{(k)}$ the jth column and the ith row of $\mathbf{A}^{k}$ respectively.

Lemma 3.1 *If* $\mathbf{A} \in \mathbb{C}^{n\times n}$ *with* $Ind\,\mathbf{A} = k$, *then*

$$\operatorname{rank}\mathbf{A}_{.i}^{k+1}\left(\mathbf{a}_{.j}^{(k)}\right) \leq \operatorname{rank}\mathbf{A}^{k+1}, \quad \forall i,j = \overline{1,n}. \tag{10}$$

Proof. The proof of this lemma is similar to that of Lemma 2.4.

In the following theorem we introduce a determinantal representation of the Drazin inverse.

Theorem 3.3 *If* $Ind\,\mathbf{A} = k$ *and* $\operatorname{rank}\mathbf{A}^{k+1} = \operatorname{rank}\mathbf{A}^{k} = r \leq n$ *for an arbitrary matrix* $\mathbf{A} \in \mathbb{C}^{n\times n}$*, then*

$$\begin{array}{l} \mathbf{A}^{D} = \left(\frac{d_{ij}}{d_r\left(\mathbf{A}^{k+1}\right)}\right)_{n\times n}, \\ \textit{where} \\ d_r\left(\mathbf{A}^{k+1}\right) = \sum\limits_{\beta\in J_{r,n}} \left|\left(\mathbf{A}^{k+1}\right)^{\beta}_{\beta}\right|, \\ d_{ij} = \sum\limits_{\beta\in J_{r,n}\{i\}} \left|\left(\mathbf{A}^{k+1}_{.i}\left(\mathbf{a}^{(k)}_{.j}\right)\right)^{\beta}_{\beta}\right|, \ \left(\forall i,j=\overline{1,n}\right). \end{array} \tag{11}$$

Proof. The proof of this theorem is analogous to that of Theorem 2.2 by using Theorem 2.1, Lemma 3.1, and Theorem 3.2.

In the following corollaries we introduce determinantal representations of the group inverse $\mathbf{A}^{\#}$ and the matrix $\mathbf{A}^{D}\mathbf{A}$ respectively.

Corollary 3.1 *If* $Ind\,\mathbf{A} = 1$ *and* $\operatorname{rank}\mathbf{A}^{2} = \operatorname{rank}\mathbf{A} = r \leq n$ *for an arbitrary matrix* $\mathbf{A} \in \mathbb{C}^{n\times n}$*, then*

$$\mathbf{A}^{\#} = \left(\frac{g_{ij}}{d_r\left(\mathbf{A}^{2}\right)}\right)_{n\times n},$$

where $d_r\left(\mathbf{A}^{2}\right) = \sum\limits_{\beta\in J_{r,n}} \left|\left(\mathbf{A}^{2}\right)^{\beta}_{\beta}\right|$, $g_{ij} = \sum\limits_{\beta\in J_{r,n}\{i\}} \left|\left(\mathbf{A}^{2}_{.i}\left(\mathbf{a}_{.j}\right)\right)^{\beta}_{\beta}\right|$, $\left(\forall i,j=\overline{1,n}\right)$.

Proof. The proof follows from Theorem 3.3 in view of $k = 1$.

Corollary 3.2 *If* $Ind\,\mathbf{A} = k$ *and* $\operatorname{rank}\mathbf{A}^{k+1} = \operatorname{rank}\mathbf{A}^{k} = r \leq n$ *for an arbitrary matrix* $\mathbf{A} \in \mathbb{C}^{n\times n}$*, then*

$$\mathbf{A}^{D}\mathbf{A} = \left(\frac{v_{ij}}{d_r\left(\mathbf{A}^{k+1}\right)}\right)_{n\times n},$$

where $v_{ij} = \sum\limits_{\beta\in J_{r,n}\{i\}} \left|\left(\mathbf{A}^{k+1}_{.i}\left(\mathbf{a}_{.j}{}^{(k+1)}\right)\right)^{\beta}_{\beta}\right|$, $\left(\forall i,j=\overline{1,n}\right)$.

Proof. Representing the Drazin inverse $\mathbf{A}^{D}$ by (11) we obtain

$$\mathbf{A}^{D}\mathbf{A} = \left(\frac{d_{ij}}{d_r\left(\mathbf{A}^{k+1}\right)}\right)_{n\times n} \cdot \left(a_{ij}\right)_{n\times n} = \left(\frac{v_{ij}}{d_r\left(\mathbf{A}^{k+1}\right)}\right)_{n\times n}.$$

Here for arbitrary $1 \leq i, j \leq n$ we have

$$v_{i\,j} = \sum_{s} \sum_{\beta \in J_{r,\,n}\{i\}} \left| \left((\mathbf{A}^{k+1})_{.\,i} (\mathbf{a}_{.\,s}^{(k)}) \right)_{\beta}^{\beta} \right| a_{s\,j} =$$

$$= \sum_{\beta \in J_{r,\,n}\{i\}} \sum_{s} \left| \left((\mathbf{A}^{k+1})_{.\,i} (\mathbf{a}_{.\,s}^{(k)} \cdot a_{s\,j}) \right)_{\beta}^{\beta} \right| = \sum_{\beta \in J_{r,\,n}\{i\}} \left| \left((\mathbf{A}^{k+1})_{.\,i} (\mathbf{a}_{.j}^{(k+1)}) \right)_{\beta}^{\beta} \right| . \blacksquare$$

Remark 3.2 *The matrix* $(\mathbf{A}^D \mathbf{A})$ *is idempotent. Similarly to the case of Remark 2.6, the matrix* $\mathbf{D} = (d_{ij})_{n \times n}$ *can be considered as an analogue of the classical adjoint matrix, where* $\mathbf{D} = d_r \left(\mathbf{A}^{k+1} \right) \mathbf{A}^D$.

4 Cramer rules for generalized inverse solutions

Definition 4.1 *Suppose in a complex system of linear equations:*

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{y} \tag{12}$$

the coefficient matrix $\mathbf{A} \in \mathbb{C}_r^{m \times n}$ *and a column of constants* $\mathbf{y} = (y_1, \ldots, y_m)^T \in \mathbb{C}^m$. *The least squares solution of the system (12) is the vector* $\mathbf{x}^0 \in \mathbb{C}^n$ *satisfying*

$$\left\| \mathbf{x}^0 \right\| = \min_{\tilde{\mathbf{x}} \in \mathbb{C}^n} \left\{ \|\tilde{\mathbf{x}}\| \mid \|\mathbf{A} \cdot \tilde{\mathbf{x}} - \mathbf{y}\| = \min_{\mathbf{x} \in \mathbb{C}^n} \|\mathbf{A} \cdot \mathbf{x} - \mathbf{y}\| \right\},$$

where $\mathbb{C}^n$ *is an* n*-dimension complex vector space.*

Theorem 4.1 *[9] The vector* $\mathbf{x} = \mathbf{A}^{+}\mathbf{y}$ *is the least squares solution of the system (12).*

Theorem 4.2 *The following statements are true for the system of linear equations (12).*

i) If $\operatorname{rank} \mathbf{A} = n$, *then the components of the least squares solution* $\mathbf{x} = (x_1^0, \ldots, x_n^0)^T$ *are obtained by the formula*

$$x_j^0 = \frac{\det(\mathbf{A}^* \mathbf{A})_{.\,j} (\mathbf{f})}{\det \mathbf{A}^* \mathbf{A}}, \quad \left(\forall j = \overline{1, n} \right), \tag{13}$$

where $\mathbf{f} = \mathbf{A}^* \mathbf{y}$.

ii) If $\operatorname{rank}\mathbf{A} = r \le m < n$*, then*

$$x_j^0 = \frac{\sum\limits_{\beta \in J_{r,n}\{j\}} \left| \left((\mathbf{A}^*\mathbf{A})_{.\,j}(\mathbf{f}) \right)_\beta^\beta \right|}{d_r\left(\mathbf{A}^*\mathbf{A}\right)}, \quad \left(\forall j = \overline{1,n}\right). \tag{14}$$

Proof. i) If $\operatorname{rank}\mathbf{A} = n$, then we can represent $\mathbf{A}^+$ by (7). By multiplying $\mathbf{A}^+$ into $\mathbf{y}$ we get (13).

ii) If $\operatorname{rank}\mathbf{A} = k \le m < n$, then $\mathbf{A}^+$ can be represented by (1). By multiplying $\mathbf{A}^+$ into $\mathbf{y}$ the least squares solution of the linear system (12) is given by components as in (14). ■

Using (3) and (8), we can obtain another representation of the Cramer rule for the least squares solution of a linear system.

Theorem 4.3 *The following statements are true for a system of linear equations written in the form* $\mathbf{x} \cdot \mathbf{A} = \mathbf{y}$*.*

i) If $\operatorname{rank}\mathbf{A} = m$*, then the components of the least squares solution* $\mathbf{x}^0 = \mathbf{y}\mathbf{A}^+$ *are obtained by the formula*

$$x_i^0 = \frac{\det(\mathbf{A}\mathbf{A}^*)_{i\,.}(\mathbf{g})}{\det \mathbf{A}\mathbf{A}^*}, \quad \left(\forall i = \overline{1,m}\right),$$

where $\mathbf{g} = \mathbf{y}\mathbf{A}^*$*.*

ii) If $\operatorname{rank}\mathbf{A} = r \le n < m$*, then*

$$x_i^0 = \frac{\sum\limits_{\alpha \in I_{r,m}\{i\}} \left| \left((\mathbf{A}\mathbf{A}^*)_{i\,.}(\mathbf{g}) \right)_\alpha^\alpha \right|}{d_r\left(\mathbf{A}\mathbf{A}^*\right)}, \quad \left(\forall i = \overline{1,m}\right).$$

Proof. The proof of this theorem is analogous to that of Theorem 4.2.

Remark 4.1 *The obtained formulas of the Cramer rule for the least squares solution differ from similar formulas in [3, 6, 10, 14, 15, 16]. They give a closer approximation to the Cramer rule for consistent nonsingular systems of linear equations.*

In some situations, however, people pay more attention to the Drazin inverse solution of singular linear systems [6, 16]. Consider a general system of linear equations (12), where $\mathbf{A} \in \mathbb{C}^{n \times n}$ and $\mathbf{x}$, $\mathbf{y}$ are vectors in $\mathbb{C}^n$. $R(\mathbf{A})$ denotes the range of $\mathbf{A}$ and $N(\mathbf{A})$ denotes the null space of $\mathbf{A}$. The characteristic of the Drazin inverse solution $\mathbf{A}^D\mathbf{y}$ is given in [16] by the following theorem.

Theorem 4.4 *Let* $\mathbf{A} \in \mathbb{C}^{n\times n}$ *with* $Ind(A) = k$*. Then* $\mathbf{A}^D\mathbf{y}$ *is both the unique solution in* $R(\mathbf{A}^k)$ *of*

$$\mathbf{A}^{k+1}\mathbf{x} = \mathbf{A}^{\mathrm{k}}\mathbf{y}, \tag{15}$$

and the unique minimal $\mathbf{P}$*-norm least squares solution of (12).*

Remark 4.2 *The* $\mathbf{P}$*-norm is defined as* $\|\mathbf{x}\|_{\mathbf{P}} = \|\mathbf{P}^{-1}\mathbf{x}\|$ *for* $\mathbf{x} \in \mathbb{C}^n$*, where* $\mathbf{P}$ *is a nonsingular matrix that transforms* $\mathbf{A}$ *into its Jordan canonical form (9).*

Remark 4.3 *Since (15) is analogous to the normal system* $\mathbf{A}^*\mathbf{A}\mathbf{x} = \mathbf{A}^*\mathbf{y}$*, the system (15) is called the generalized normal equations of (12), (see [16]).*

We obtain the Cramer rule for the $\mathbf{P}$-norm least squares solution of (12) in the following theorem.

Theorem 4.5 *Let* $\mathbf{A} \in \mathbb{C}^{n\times n}$ *with* $Ind(\mathbf{A}) = k$*. Then the unique minimal* $\mathbf{P}$*-norm least squares solution* $\widehat{\mathbf{x}} = (\widehat{x}_1, \dots, \widehat{x}_n)^T$ *of the system (12) is given by*

$$\widehat{x}_i = \frac{\sum\limits_{\beta \in J_{r,\,n}\{i\}} \left| \left(\mathbf{A}^{k+1}_{.\,i}\left(\mathbf{g}\right)\right)^{\beta}_{\beta} \right|}{\sum\limits_{\beta \in J_{r,n}} \left| \left(\mathbf{A}^{k+1}\right)^{\beta}_{\beta} \right|} \quad \forall i = \overline{1,n}, \tag{16}$$

where $\mathbf{g} = \mathbf{A}^k\mathbf{y}$.

Proof. Representing the Drazin inverse by (11) and by virtue of Theorem 4.4, we have

$$\widehat{\mathbf{x}} = \begin{pmatrix} \widehat{x}_1 \\ \dots \\ \widehat{x}_n \end{pmatrix} = \mathbf{A}^D\mathbf{y} = \frac{1}{d_r\left(\mathbf{A}^{k+1}\right)} \begin{pmatrix} \sum\limits_{s=1}^{n} d_{1s}y_s \\ \dots \\ \sum\limits_{s=1}^{n} d_{ns}y_s \end{pmatrix}.$$

Therefore,

$$\widehat{x}_i = \frac{1}{d_r\left(\mathbf{A}^{k+1}\right)} \sum_{s=1}^{n} \sum_{\beta \in\, J_{r,n}\{i\}} \left| \left(\mathbf{A}^{k+1}_{.\,i}\left(\mathbf{a}^{(k)}_{.\,s}\right)\right)^{\beta}_{\beta} \right| \cdot y_s =$$

$$= \frac{1}{d_r\left(\mathbf{A}^{k+1}\right)} \sum_{\beta \in J_{r,\,n}\{i\}} \sum_{s=1}^{n} \left| \left(\mathbf{A}_{.i}^{k+1}\left(\mathbf{a}_{.s}^{(k)}\right)\right)_{\beta}^{\beta} \right| \cdot y_s =$$

$$= \frac{1}{d_r\left(\mathbf{A}^{k+1}\right)} \sum_{\beta \in J_{r,\,n}\{i\}} \sum_{s=1}^{n} \left| \left(\mathbf{A}_{.i}^{k+1}\left(\mathbf{a}_{.s}^{(k)} \cdot y_s\right)\right)_{\beta}^{\beta} \right|.$$

From this (16) follows immediately. ■

5 Examples

1. Let us consider the system of linear equations.

$$\begin{cases} 2x_1 - 5x_3 + 4x_4 = 1, \\ 7x_1 - 4x_2 - 9x_3 + 1.5x_4 = 2, \\ 3x_1 - 4x_2 + 7x_3 - 6.5x_4 = 3, \\ x_1 - 4x_2 + 12x_3 - 10.5x_4 = 1. \end{cases} \tag{17}$$

The coefficient matrix of the system is the matrix $\mathbf{A} = \begin{pmatrix} 2 & 0 & -5 & 4 \\ 7 & -4 & -9 & 1.5 \\ 3 & -4 & 7 & -6.5 \\ 1 & -4 & 12 & -10.5 \end{pmatrix}$.

We calculate the rank of $\mathbf{A}$ which is equal to 3, and we have

$$\mathbf{A}^* = \begin{pmatrix} 2 & 7 & 3 & 1 \\ 0 & -4 & -4 & -4 \\ -5 & -9 & 7 & 12 \\ 4 & 1.5 & -6.5 & -10.5 \end{pmatrix}, \mathbf{A}^*\mathbf{A} = \begin{pmatrix} 63 & -44 & -40 & -11.5 \\ -44 & 48 & -40 & 62 \\ -40 & -40 & 299 & -205 \\ -11.5 & 62 & -205 & 170.75 \end{pmatrix}.$$

At first we obtain entries of $\mathbf{A}^+$ by (1):

$$d_3(\mathbf{A}^*\mathbf{A}) = \begin{vmatrix} 63 & -44 & -40 \\ -44 & 48 & -40 \\ -40 & -40 & 299 \end{vmatrix} + \begin{vmatrix} 63 & -44 & -11.5 \\ -44 & 48 & 62 \\ -11.5 & 62 & 170.75 \end{vmatrix} +$$

$$+ \begin{vmatrix} 63 & -40 & -11.5 \\ -40 & 299 & -205 \\ -11.5 & -205 & 170.75 \end{vmatrix} + \begin{vmatrix} 48 & -40 & 62 \\ -40 & 299 & -205 \\ 62 & -205 & 170.75 \end{vmatrix} = 102060,$$

$$l_{11} = \begin{vmatrix} 2 & -44 & -40 \\ 0 & 48 & -40 \\ -5 & -40 & 299 \end{vmatrix} + \begin{vmatrix} 2 & -44 & -11.5 \\ 0 & 48 & 62 \\ 4 & 62 & 170.75 \end{vmatrix} + \begin{vmatrix} 2 & -40 & -11.5 \\ -5 & 299 & -205 \\ 4 & -205 & 170.75 \end{vmatrix} =$$

$$= 25779,$$

and so forth. Continuing in the same way, we get

$$\mathbf{A}^{+}=\frac{1}{102060}\begin{pmatrix}25779 & -4905 & 20742 & -5037\\ -3840 & -2880 & -4800 & -960\\ 28350 & -17010 & 22680 & -5670\\ 39558 & -18810 & 26484 & -13074\end{pmatrix}.$$

Now we obtain the least squares solution of the system (17) by the matrix method.

$$\mathbf{x}^{0}=\begin{pmatrix}x_{11}^{0}\\ x_{21}^{0}\\ x_{31}^{0}\\ x_{41}^{0}\end{pmatrix}=\frac{1}{102060}\begin{pmatrix}25779 & -4905 & 20742 & -5037\\ -3840 & -2880 & -4800 & -960\\ 28350 & -17010 & 22680 & -5670\\ 39558 & -18810 & 26484 & -13074\end{pmatrix}\cdot\begin{pmatrix}1\\ 2\\ 3\\ 1\end{pmatrix}=$$

$$=\frac{1}{102060}\begin{pmatrix}73158\\ -24960\\ 56700\\ 68316\end{pmatrix}=\begin{pmatrix}\frac{12193}{17010}\\ -\frac{416}{1071}\\ \frac{5}{9}\\ \frac{5693}{8505}\end{pmatrix}.$$

Next we get the least squares solution of the system (17) by the Cramer rule (14), where

$$\mathbf{f}=\begin{pmatrix}2 & 7 & 3 & 1\\ 0 & -4 & -4 & -4\\ -5 & -9 & 7 & 12\\ 4 & 1.5 & -6.5 & -10.5\end{pmatrix}\cdot\begin{pmatrix}1\\ 2\\ 3\\ 1\end{pmatrix}=\begin{pmatrix}26\\ -24\\ 10\\ -23\end{pmatrix}.$$

Thus we have

$$x_{1}^{0}=\frac{1}{102060}\left(\begin{vmatrix}26 & -44 & -40\\ -24 & 48 & -40\\ 10 & -40 & 299\end{vmatrix}+\begin{vmatrix}26 & -44 & -11.5\\ -24 & 48 & 62\\ -23 & 62 & 170.75\end{vmatrix}+\right.$$

$$\left.+\begin{vmatrix}26 & -40 & -11.5\\ 10 & 299 & -205\\ 23 & -205 & 170.75\end{vmatrix}\right)=\frac{73158}{102060}=\frac{12193}{17010};$$

$$x_{2}^{0}=\frac{1}{102060}\left(\begin{vmatrix}63 & 26 & -40\\ -44 & -24 & -40\\ -40 & 10 & 299\end{vmatrix}+\begin{vmatrix}63 & 26 & -11.5\\ -44 & -24 & 62\\ -11.5 & -23 & 170.75\end{vmatrix}+\right.$$

$$+\begin{vmatrix} -24 & -40 & 62 \\ 10 & 299 & -205 \\ -23 & -205 & 170.75 \end{vmatrix}\Bigg) = \frac{-24960}{102060} = -\frac{416}{1071};$$

$$x_3^0 = \frac{1}{102060}\Bigg(\begin{vmatrix} 63 & -44 & 26 \\ -44 & 48 & -24 \\ -40 & -40 & 10 \end{vmatrix} + \begin{vmatrix} 63 & 26 & -11.5 \\ -40 & 10 & -205 \\ -11.5 & -23 & 170.75 \end{vmatrix} +$$

$$+\begin{vmatrix} 48 & -24 & 62 \\ -40 & 10 & -205 \\ 62 & -23 & 170.75 \end{vmatrix}\Bigg) = \frac{56700}{102060} = \frac{5}{9};$$

$$x_4^0 = \frac{1}{102060}\Bigg(\begin{vmatrix} 63 & -44 & 26 \\ -44 & 48 & -24 \\ -11.5 & 62 & -23 \end{vmatrix} + \begin{vmatrix} 63 & -40 & 26 \\ -40 & 299 & 10 \\ -11.5 & -205 & -23 \end{vmatrix} +$$

$$+\begin{vmatrix} 48 & -40 & -24 \\ -40 & 299 & 10 \\ 62 & -205 & -23 \end{vmatrix}\Bigg) = \frac{68316}{102060} = \frac{5693}{8505}.$$

2. Let us consider the following system of linear equations.

$$\begin{cases} x_1 - x_2 + x_3 + x_4 = 1, \\ x_2 - x_3 + x_4 = 2, \\ x_1 - x_2 + x_3 + 2x_4 = 3, \\ x_1 - x_2 + x_3 + x_4 = 1. \end{cases} \tag{18}$$

The coefficient matrix of the system is the matrix $\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & 2 \\ 1 & -1 & 1 & 1 \end{pmatrix}$.

It is easy to verify the following:

$$\mathbf{A}^2 = \begin{pmatrix} 3 & -4 & 4 & 3 \\ 0 & 1 & -1 & 0 \\ 4 & -5 & 5 & 4 \\ 3 & -4 & 4 & 3 \end{pmatrix}, \ \mathbf{A}^3 = \begin{pmatrix} 10 & -14 & 14 & 10 \\ -1 & 2 & -2 & -1 \\ 13 & -18 & 18 & 13 \\ 10 & -14 & 14 & 10 \end{pmatrix},$$

and $\operatorname{rank}\mathbf{A} = 3$, $\operatorname{rank}\mathbf{A}^2 = \operatorname{rank}\mathbf{A}^3 = 2$. This implies $k = Ind(\mathbf{A}) = 2$. We obtain entries of $\mathbf{A}^D$ by (11).

$$d_2(\mathbf{A}^3) = \begin{vmatrix} 10 & -14 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 10 & 14 \\ 13 & 18 \end{vmatrix} + \begin{vmatrix} 10 & 10 \\ 10 & 10 \end{vmatrix}$$

$$+\begin{vmatrix} 2 & -2 \\ -18 & 18 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -14 & 10 \end{vmatrix} + \begin{vmatrix} 18 & 13 \\ 14 & 10 \end{vmatrix} = 8,$$

$$d_{11} = \begin{vmatrix} 3 & -14 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 14 \\ 4 & 18 \end{vmatrix} + \begin{vmatrix} 3 & 10 \\ 3 & 10 \end{vmatrix} = 4,$$

and so forth. Continuing in the same way, we get $\mathbf{A}^D = \begin{pmatrix} 0.5 & 0.5 & -0.5 & 0.5 \\ 1.75 & 2.5 & -2.5 & 1.75 \\ 1.25 & 1.5 & -1.5 & 1.25 \\ 0.5 & 0.5 & -0.5 & 0.5 \end{pmatrix}$.

Now we obtain the Drazin inverse solution $\widehat{\mathbf{x}}$ of the system (18) by the Cramer rule (16), where

$$\mathbf{g} = \mathbf{A}^2\mathbf{y} = \begin{pmatrix} 3 & -4 & 4 & 3 \\ 0 & 1 & -1 & 0 \\ 4 & -5 & 5 & 4 \\ 3 & -4 & 4 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ -1 \\ 13 \\ 10 \end{pmatrix}.$$

Thus we have

$$\widehat{x}_1 = \frac{1}{8}\left(\begin{vmatrix} 10 & -14 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 10 & 14 \\ 13 & 18 \end{vmatrix} + \begin{vmatrix} 10 & 10 \\ 10 & 10 \end{vmatrix} \right) = \frac{1}{2},$$

$$\widehat{x}_2 = \frac{1}{8}\left(\begin{vmatrix} 10 & 10 \\ -1 & -1 \end{vmatrix} + \begin{vmatrix} -1 & -2 \\ 13 & 18 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 10 & 10 \end{vmatrix} \right) = 1,$$

$$\widehat{x}_3 = \frac{1}{8}\left(\begin{vmatrix} 10 & 10 \\ 13 & 13 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -18 & 13 \end{vmatrix} + \begin{vmatrix} 13 & 13 \\ 10 & 10 \end{vmatrix} \right) = 1,$$

$$\widehat{x}_4 = \frac{1}{8}\left(\begin{vmatrix} 10 & 10 \\ 10 & 10 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -14 & 10 \end{vmatrix} + \begin{vmatrix} 18 & 13 \\ 14 & 10 \end{vmatrix} \right) = \frac{1}{2}.$$

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